A cute lemma in homological algebra

October 2022

An important concept in homological algebra is that of a deformation retract. Let $(A^\bullet, d_A)$ and $(B^\bullet, d_B)$ be a pair of chain complexes. We are working cohomologically, so that the differentials raise the degrees by 1. A deformation retract is the following data:

- a chain map $i : (A^\bullet, d_A) \to (B^\bullet, d_B)$,
- a chain map $p : (B^\bullet, d_B) \to (A^\bullet, d_A)$,
- and a degree -1 map $h : B^\bullet \to B^{\bullet -1}$,

such that $p \circ i = id_A$, and $id_B - i \circ p = d_B \circ h + h \circ d_B$. Intuitively, we can think of h as arising from a deformation retraction from a space whose cohomology is modelled by $(B^\bullet, d_B)$ to a subspace whose cohomology is modelled by $(A^\bullet, d_A)$. Often, the so-called 'side conditions' are also required to hold:

$$h \circ i = 0, \qquad p \circ h = 0, \qquad h \circ h = 0.$$

Let us call the collection of complexes and maps $(A^\bullet, B^\bullet, d_A, d_B, i, p, h)$, satisfying the side conditions, as an HR package.

One upshot of this data is that i and p are quasi-isomorphisms: they induce isomorphisms between the cohomology groups $H^\bullet(A)$ and $H^\bullet(B)$. The fact that we have the explicit homotopy, rather than just the knowledge that i and p are quasi-isomorphisms, can often come in handy. For example, if we now deform d_B to a new differential $d_B + \epsilon$, then the homological perturbation lemma tells us that if ϵ is small enough in a precise sense, then the full HR package can be deformed along with d_B. In fact, this lemma gives us the precise recipe for the deformation of the HR package: $(A^\bullet, B^\bullet, d_{A,\epsilon}, d_B + \epsilon, i_\epsilon, p_\epsilon, h_\epsilon)$. In practice this can be extremely useful.

But I don't want to talk about that now. Instead, I want to talk about the very special case of 2-term chain complexes. So from now on, assume that our chain complexes have the form

$$d_A : A^0 \to A^1, \qquad d_B : B^0 \to B^1.$$

Let's also assume to begin with that we only have the chain map $i : A^\bullet \to B^\bullet$. We want to somehow produce the rest of the HR package with as little effort as possible. Let's phrase this as a problem.

Question 1. Suppose that i is a quasi-isomorphism. How do we produce the rest of the HR package?

The first thing to note is that such an HR package involves p satisfying $p \circ i = id_A$. So there's no way this is going to work unless i is injective. So let's assume this as well.

Now it might seem that producing the rest of the data involves a few choices: we are going to have to choose maps $p_0 : B^0 \to A^0$, $p_1 : B^1 \to A^1$, and $h : B^1 \to B^0$. And we need to choose these in a rather delicate way, so that all the equations of the HR package are satisfied. It is certainly possible to do this, but it's not so clear what choices are involved. What I want to explain is the following lemma:

Lemma 1. *Let $p_0 : B^0 \to A^0$ be a splitting of the map $i_0 : A^0 \to B^0$. Then there are canonically determined maps p and h defining an HR package, such that p_0 is the chosen map.*

The cool thing is that we only need to make one choice of map, and check that it satisfies one equation. Everything else is determined. The key to doing this is another very useful construction from homological algebra: the mapping cone. In fact, we only need a baby version of this. Given the two complexes $(A^\bullet, d_A)$ and $(B^\bullet, d_B)$, and the map i, we can form the following 3-term complex:

$$A^0 \to A^1 \oplus B^0 \to B^1,$$

where the first map is given by $a(x) = (d_A(x), i_0(x))$, and the second map is given by $b(y, z) = i_1(y) - d_B(z)$. The fact that this is a chain complex just reflects the fact that i is a chain map. I'll leave the following lemma as an exercise.

Lemma 2. *The chain map i is a quasi-isomorphism if and only if the above 3-term complex defines a short exact sequence.*

Let's now go ahead and prove Lemma 1. So we assume that i is an injective quasi-isomorphism, and we start with a map $p_0 : B^0 \to A^0$, such that $p_0 \circ i_0 = id_{A^0}$. We now use this to define a splitting of the above short exact sequence. Namely, define

$$t : A^1 \oplus B^0 \to A^0, \qquad (y, z) \mapsto p_0(z).$$

You can easily check that $t \circ a = id_{A^0}$. But now, since we have a short exact sequence, the map t automatically determines a splitting s of b. More precisely, there is a uniquely defined map

$$s : B^1 \to A^1 \oplus B^0, \qquad w \mapsto (p_1(w), -h(w)),$$

such that

- $b \circ s = id_{B^1}$, and
- $a \circ t + s \circ b = id_{A^1 \oplus B^0}$.

It also immediately follows that $t \circ s = 0$. So we get an exact sequence in the opposite direction, and maps $p_1 : B^1 \to A^1$ and $h : B^1 \to B^0$. These are the other maps that we had to find!

To finish the proof, let's now unpack the three identities and see that they imply the HR equations.

1. From $(y, 0) = at(y, 0) + sb(y, 0)$ we get

   $$(y, 0) = (p_1 i_1(y), -h i_1(y)).$$

   This gives us $p_1 \circ i_1 = id_{A^1}$. Hence combining with p_0, we have our map p which satisfies $p \circ i = id_A$. And $h \circ i_1 = 0$ is one of the side conditions.

2. From $(0, z) = at(0, z) + sb(0, z)$ we get

   $$(0, z) = (d_A p_0(z) - p_1 d_B(z), i_0 p_0(z) + h d_B(z)).$$

   This tells us that p is a chain map and that $id_{B^0} - i_0 p_0 = h d_B$, which is half of the homotopy equation.

3. From $b \circ s = id_{B^1}$ we get $id_{B^1} - i_1 p_1 = d_B h$, the other half of the homotopy equation. Hence we have $id_B - i \circ p = d_B \circ h + h \circ d_B$.

4. And from $t \circ s = 0$ we get $p_0 \circ h = 0$, another side condition.

The missing side condition $h \circ h = 0$ follows automatically for degree reasons.